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BALANCING PAIRS OF INTERFERING ELEMENTS

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1. Introduction

Many decisions in different fields of application have to take into account the joint effects of two elements that can interfere with each other. For example, in Industrial Economics the demand for an asset can be influenced by the supply of another asset, with synergic or antagonistic effects. The same happens in Public Economics, where two differing economic policies can create mutual interference.

Analogously, the same is true about drugs in Veterinary Science and Medicine, additives in Agriculture, diets in Zoo-technics and so on. When it is necessary to use such elements, there is sometimes a primary interest in one effect rather than the other: for instance, the effect of one could be twice as strong as that of the other. In such cases, it is important to consider the extent of influence of the elements while deciding about the dosage of them.

In this paper we provide a direct method, not an iterative one, to obtain the optimal solution of the problem.

In the next section, we shall give the basic definitions and in the next three ones we will present the optimization problem and the solution method, together with its theoretical foundation. Some applications to Economics and other fields are presented in sections 6 and 7.

2. Definitions

Let $N = \{1, 2\}$ be the set of labels of the two interfering elements (commodities, or drugs, feed and so on) and of the related effects resulting from their use (commodity demand, states of health or nutrition, and so on). From here on, if not otherwise specified, use of index i will imply "for all $I \in N$", with an analogous use of index j.

We denote the non-negative quantities of the i-th element as follows:

- Q_i is the quantity effectively used;
- Q_i^{max} is the optimal quantity if the i-th element is used alone;
- Q_i^{min} is the minimum necessary quantity if the i-th element is used alone;
- q_i and q_i^{min} are the corresponding ratios with respect to Q_i^{max}:
  - $q_i = Q_i/Q_i^{max}$,
  - $q_i^{min} = Q_i^{min}/Q_i^{max}$.

It is assumed that $Q_i^{min} < Q_i^{max}$ and $Q_i^{min} \leq Q_i \leq Q_i^{max}$.

Given such conditions, q_i and q_i^{min} belong to interval [0,1].

We call Q, Q^{max}, Q^{min}, q, and q^{min} the corresponding n-vectors.

Let $e_i(q)$ be a non-negative function expressing the level of the i-th effect when percentage quantities q are used. The space of the effects is the set of points $x = (x_1, ..., x_n) = e(q)$ according to variations of q. This function should satisfy the following conditions (which should be present, given a suitable adjustment of scale).

If no elements are used, then all the effects are null. If a single element is employed in the optimal dose for using it alone, then the level of the relative effect is 1, while the level of the effect for the other is null. Finally, if both elements are employed in the optimal doses for separate usage, the resulting effects are given by vector $\boldsymbol{\delta} = (\delta_1, \delta_2)$ with real positive components. In formulae:

- if $q_i = 0$ for all $i \in N$, then $e_i = 0$ for all $i \in N$;
- if $q_i = 1$ and $q_j = 0$ for all $j \in N, j \neq i$, then $e_i = 1$ and $e_j = 0$;
- if $q_i = 1$ for all $i \in N$, then $e_i = \delta_i$.

Without loss of generality, we may sort the elements so that:

$$\delta_1 \leq \delta_2 \tag{1}$$

We use e_i^{min} to indicate the minimum necessary level of the i-th effect. This level is derived from function $e_i(q)$ given $q_i = q_i^{min}$ and $q_j = 0$ for the other component $j \neq i$. We use e^{min} to indicate the related n-dimensional vector.

We assume that the minimum necessary level of the i-th effect should not exceed 1 (if $\delta_i \leq 1$) or δ_i (elsewhere). Thus

$$e_i^{\min} \leq \max\{1, \delta_i\} \tag{2}$$

Importing a classical definition, we will call every point x of the codomain of e which is not jointly improvable a *Pareto optimal effect*, in the sense that if we move from that point in this set to improve the i-th effect, then the other effect necessarily decreases. It is easy to prove that even here every Pareto optimal point is a boundary point of the set of effects; we shall therefore call the set of Pareto optimal effects the *Pareto optimal boundary.*

The term *feasible Pareto optimal boundary P* is given to the set of the points of the Pareto optimal boundary that respect the conditions $x_i \geq e_i^{\min}$ for all $i \in N$.

We use r to indicate the required optimal ratio between the effects e_1 and e_2.

We call R the half-line centred on the origin, the slope of which is defined by r.

For each point x of the feasible set, we use E to indicate the half-line centred on the origin, passing through x.

3. The optimization problem

The input data of the model are δ, $e^{\min}$ and r.

In some applications we do not know directly the minimal effect $e_i^{\min}$ for some element i, while we know the necessary minimal and optimal quantities $Q_i^{\min}$ and $Q_i^{\max}$. From this relationship, it is thus possible to deduce $q_i^{\min}$ which, introduced into the equation $e_i(q)$, gives $e_i^{\min}$.

The problem is to find the set of vectors q^* such that the corresponding effects $e(q^*)$ belong to the feasible Pareto optimal boundary and are such that the half-lines that join them to the origin form a minimum angle with R.

If the necessary minimum effects are excessive as a whole, the feasible set might possibly be empty and therefore the problem is without a solution. However, for those cases where determination of the minimum quantity is open to variations, we have introduced certain indications as well as modifications to be used each time. The uniqueness of the solution should also be checked each time.

4. Solution method

Determination of the optimal combination of q clearly depends on the form of the effects function $e(q)$. This function may be defined directly, according to the type of problem, or may be constructed on the basis of available cases, using statistical methods. In any case, for the vertices of the domain, the values determined in section 2 should be respected (to obtain suitable conditions, it would be possible to use, for example, an adjustment of scale).

Assuming that the effects' functions are continuous, the following approach may be used. Wherever possible, this approach helps to obtain explicit equations for the Pareto optimal boundary, thereby avoiding any need to resort to numeric methods, which may be unable to guarantee precision in results. Moreover, explicit equations make further analysis much easier to be carried out.

The method we are about to present is based on a theorem that will be introduced in section 5.

Thus let $e(q)$ be a continuous function that respects constraint (1).

We begin by examining the function in the interior of the domain. Here the "critical zone" must be determined, that is, the set of points (q_1, q_2) in which the function may be differentiated and which render to zero the determinant of the matrix of the first partial derivatives (Jacobian) of $e(q)$. Let I indicate the image of $e(q)$ in the critical zone. If I is not empty, it is characterised by h functions $\underline{I}_s$ $(s = 1, \ldots, h)$ defined on the space of the effects.

In the event of I being empty, there are no functions to characterise it and we shall see how to proceed further on.

Let B be the image of the function on the boundary, that is, on the four sides of the quadrangle that makes up the domain. B is characterised by four functions $\underline{B}_s$ ($s = 1, \ldots, 4$) defined on the space of the effects.

Let us now consider the set of points in the interior of the domain, in which the function may not be differentiable; we use A to indicate this set. If set $e(A)$ can be characterised by a finite number w of functions defined on the space of the effects, let us call these functions $\underline{W}_s$ ($s = 1, \ldots, w$); otherwise, we shall use the approach described below at point b.

In the first case, it can be proved (see below) that the Pareto optimal boundary belongs to $I \cup B \cup e(A)$.

Let us use g to denote an indexed family of functions $\underline{I}_s$, $\underline{B}_s$ and $\underline{W}_s$ (if, for instance, the functions are $\underline{I}_1$, $\underline{I}_2$, $\underline{B}_1$, $\underline{B}_2$, $\underline{B}_3$, $\underline{B}_4$, $\underline{W}_1$, then a family g can be made up of: $g_1 = \underline{B}_1$, $g_2 = \underline{I}_1$, $g_3 = \underline{W}_1$, $g_4 = \underline{B}_2$, $g_5 = \underline{I}_2$, $g_6 = \underline{B}_3$, $g_7 = \underline{B}_4$).

For each function g_i we use $D(g_i)$ to indicate the relative domain.

There are two possibilities:

a) I is not empty and all functions $\underline{I}_s$, $\underline{B}_s$ and $\underline{W}_s$ can be solved for x_2;

b) other cases.

In case (a) the Pareto optimal boundary is the set of $x = (x_1, x_2(x_1))$ such that:

$$\begin{cases} x_2(x_1) = \max\limits_{i \in U(x_1)} (g_i(x_1)) \\ U(x_1) = \{i : x_1 \in D(g_i)\} \end{cases}.$$

In cases (b) the Pareto optimal boundary can be identified by means of a graph representing the set $I \cup B \cup e(A)$ (this will be illustrated below with an example).

Let us now calculate the intersection between the half-line R and the feasible Pareto optimal boundary P. If such an intersection exists, it is unique (for reasons of Pareto-optimality) and in this case the solution to the problem is the pair (q_1, q_2) that corresponds to this point.

If such an intersection does not exist, then we need to solve the optimization problem:

$$\min_{q_1, q_2} \left| \frac{e_2(q)}{e_1(q)} - r \right|$$
$$s.t.(e_1(q), e_2(q)) \in P.$$

The above optimization problem can be solved, in closed form, by the same principle of the main theorem, taking into account that we have a *simple parametric form* $s : I \to P$ of the Pareto boundary P. Briefly, our optima q can be found by considering the function:

$$h(X,Y) := |Y/X - r|.$$

The optima of the function $h°s$ on the interval I (which determine immediately the optima of h on P) are located on the two border points of I, or on the stationary points of the compose function $h°s$ or on the points of I where the compose function $h°s$ is not differentiable or it is not continuous. Then, a simple comparison among a finite number of points concludes the search of the minima of $h°s$ and thus of h itself. The solution to the initial optimization problem is thus made up of all the pairs $q = (q_1, q_2)$, whose corresponding values $e_1(q)$ e $e_2(q)$ are the minima of function h.

5. Theoretical Foundations

The method proposed above was obtained by adapting the method given by Carfĕ (2009" 38–42) to the above described situation. The method deals only with differentiable functions. The complete proof for cases of continuous functions is given by the following.

Theorem. *Let f be a function defined on a compact subset K of the Euclidean plane and taking values into the same plane. Let ∂K be the topological boundary of K; let C be the set of all interior points of K where the function f is differentiable and the Jacobian matrix of f is not invertible; let H be the set of all the interior points of K in which f is not differentiable (note that this set must contain the set of all interior points of K in which f is not continuous). Then, the part of the boundary of the image of the compact K which is contained in the image f(K), that is the set* $\partial f(K) \cap f(K)$*, is contained in the union of the following three sets: the image of the boundary of the compact K, that is the set* $f(\partial K)$*; the image of the interior critical zone C of the function f, that is the image* $f(C)$*; the image of the non-differentiable zone H, that is* $f(H)$*. In particular, the Pareto Optimal boundary of the image* $f(K)$ *is contained in the above union.*

Proof. The intersection $\partial f(K) \cap f(K)$ is the image $f(K)$ minus the interior part of $f(K)$. In other words, a point x of the image $f(K)$ is a boundary point if and only if it is not an interior point. Moreover, the intersection $\partial f(K) \cap f(K)$ is obviously contained in the image $f(K)$. So the intersection $\partial f(K) \cap f(K)$ is contained in $f(K°) \cup f(\partial K)$, where $K°$ is the interior part of the compact K. Moreover the image $f(K°)$ is the union $f(H) \cup f(K° \setminus H)$. So the intersection $\partial f(K) \cap f(K)$ is contained in the union $f(\partial K) \cup f(H) \cup f(K° \setminus H)$.

More specifically, the part $f(K° \setminus H)$ is contained in the union of the two parts $f(C)$ and $f(K° \setminus (H \cup C))$; but the part $f(K° \setminus (H \cup C))$ contains only interior points of $f(K)$, which cannot be boundary points. So we can conclude that the intersection $\partial f(K) \cap f(K)$ is contained in the union $f(\partial K) \cup f(H) \cup f(C)$.

Notice that the part $f(K° \setminus (H \cup C))$ contains only interior points since the function f is a local homeomorphism at every point x belonging to the subset $K° \setminus (H \cup C)$. Indeed, this latter difference set is the set of all interior points of K in which the function f is differentiable and with invertible Jacobian matrix; hence f is a local diffeomorphism at these points. As we already know, a local diffeomorphism at a point x is also a local homeomorphism at that point, so that it sends a neighbourhood of x into a neighbourhood of $f(x)$; consequently $f(x)$ is also an internal point.

Q.E.D.

6. Some applications to Economics

The problem of finding the optimal quantities of goods to be produced is well-known. The fact that the demand for certain goods can be influenced by an interaction with the demand for other goods often plays a part in this problem. In some cases a firm has to decide the production quantities of a product that can partially or completely substitute other products ("cannibalization"). In other cases the effects of two products can be synergic (complementary).

Let us consider, as an example, the case of a company producing a particular commodity (denoted by A), but which has just developed a new commodity (denoted by B), the demand for which might negatively influence the demand for A. We assume that the company does not want to produce in order to create warehouse stock.

In the first place, the company has to calculate the optimal quantity it would sell when marketing only A (Q_A^{max}), and similarly only B (Q_B^{max}). Obviously, it could decide to sell no products at all, thereby rendering the quantities Q_A^{min} and Q_B^{min} equal to zero.

Let $e_A(q_A, q_B)$ be the projected market demand for product A, given the hypothesis in which percentage quantities q_A for A and q_B for B are sold. Thus δ_A and δ_B are measured by the respective demand for products *A* and *B* in the case where both products are sold in quantities Q_A^{max} and Q_B^{max}.

The decision regarding the quantities of B to sell depends on a willingness to sacrifice part of the demand for A. This willingness to cannibalize product A depends on various factors, examples being the future market situation of the two products and a company's desire to place itself at a strategic advantage in an emerging market (the one for B); for a detailed analysis of the factors influencing the willingness to cannibalize see Chandy et al. (1998), Nijssen et al. (2004) and Battaggion et. al. (2009).

Thus, the willingness to cannibalize is represented by *r*, the desired trade-off between demand for one product and demand for the other.

With the problem defined in these terms, the company can calculate the optimal quantities to produce, applying the methods provided in the previous sections.

7. Some other applications

The model can be used in the same way in Public Economics to calibrate two differing economic policies that are interfering with each other. There are also other applications outside economics.

In Medicine and Veterinarian practice the balance of interfering drugs is usually determined by successive approximations, keeping the patient monitored. Thus the decision on the first dose is particularly delicate. Using this model, it is possible to establish the optimal doses in relation to the desired ratios between improvements in the patient's health with respect to two diseases, taking into account the necessary minimal quantity of each medicine.

The model can be also applied in Zootechnics to optimize diets, in Agriculture to calculate doses of parasiticides or additives, so as to increase production, and so on.

8. Further developments

There are many concrete applications of this model to real life. Further outstanding problems are methods for non-continuous cases, which have not been resolved here, and new interpretations shifting from a Decision Theory viewpoint to that of Game Theory (see for instance Fragnelli and Gambarelli (2013a and 2013b).

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Summary

Many decisions in different fields of application have to take into account the joint effects of two elements that can interfere with each other. For example, in Industrial Economics the demand for an asset can be influenced by the supply of another asset, with synergic or antagonistic effects. The same happens in Public Economics, where two differing economic policies can create mutual interference. Analogously, the same can be said about drugs in Veterinary Science and Medicine, additives in agriculture, diets in zoo-technics and so on. When it is necessary to use such elements, there is sometimes a primary interest for one effect rather than another: for instance, the effect/influence of one could be twice as large as that of the other. In such cases, it is important to consider the extent of influence of the elements in the dose of the elements.

With this in mind, the model proposed here helps to determine optimal quantities of two elements that interfere with each other, taking into account the minimum quantities to be allocated. A method for determining solutions for continuous effects' functions is given. The specific quality of this model is that it provides a direct method, and not an iterative one, to obtain the solution.

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Streszczenie

Wiele decyzji z różnych dziedzin musi uwzględniać wzajemne oddziaływanie dwóch czynników, które mogą ze sobą kolidować. Na przykład, w ujęciu rynkowym, na popyt na pewne aktywa może wpływać podaż innych aktywów, dając efekt synergiczny lub antagonistyczny. Podobnie dzieje się w sferze publicznej, gdzie dwie różne polityki gospodarcze mogą być ze sobą sprzeczne. Analogicznie, to samo można powiedzieć o lekach w medycynie lub weterynarii, nawozach w rolnictwie, diecie w zoo-technice i tak dalej. Gdy konieczne jest stosowanie takich czynników, czasami głównie interesuje nas jeden efekt, a nie drugi: na przykład, efekt/wpływ jednego czynnika może być dwukrotnie większy niż drugiego. W takich przypadkach ważne jest, aby wziąć pod uwagę wielkość wpływu czynnika przy jego określonej ilości.

Mając to na uwadze, zaproponowano model, który pomaga ustalić optymalne ilości dwóch czynników, które zakłócają się wzajemnie, przy uwzględnieniu minimalnych ilości każdego z nich.

Zaprezentowana metoda znajduje rozwiązania dla ciągłej funkcji celu. Zaletą tego modelu jest to, że zapewnia bezpośredni, a nie iteracyjny, sposób rozwiązania.